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The final approach to steady state in nonsteady stagnation point heat transfer

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SUMMARY

The thermal response of a laminar boundary layer flow near a stagnation point due to step wall temperature change is investigated when the elapsed time is large. The final approach to the steady state temperature field is shown to be characterized by exponential decay with time of perturbations from the steady state. The characteristic factors appearing in the exponents arise from the solution of an eigenvalue problem in ordinary linear differential equations. Results are presented for Prandtl numbers of 0.01 to 100 for two dimensional stagnation flow and 0.1 to 10 for axisymmetrical stagnation flow.

Nomenclature

u	velocity component in the x-direction
v	velocity component in the y-direction
x	distance from the front stagnation, along the contour for two-dimensional boundary layer
y	distance perpendicular to the surface
T	temperature
t	time
ν	kinematic viscosity
α	thermal diffusivity
ξ	dimensionless coordinate, defined in (11)
η	dimensionless coordinate, defined in (11)
f	function definined by (12)
θ	dimensionless temperature defined by (16)
τ	dimensionless time defined by (17)
Pr	Prandtl number
K	proportionality constant. $K = 2U_\infty/R$ for a cylinder and $3U_\infty/(2R)$ for a sphere; U_∞ is the free stream velocity and R the radius
k	thermal conductivity
$1(t)$	Heaviside unit operator $= 0$ for $t < 0$ and $= 1$ for $t \geqq 0$
λ_1	first eigenvalue
λ_2	second eigenvalue
ψ	stream function defined by (10)
$\Phi(\eta)$	eigenfunction defined by (24)

Γ gamma function
$a = f''(0)$ = 1.2326 for two-dimensional stagnation and 0.9277 for axisymmetrical stagnation
q instantaneous heat flux
$\bar{\theta}(\eta, p)$ Laplace transform of $\theta(\eta, \tau)$

Subscripts
ss steady state
w conditions at wall
∞ free stream condition

1. Introduction

In a series of papers [1–3], the non-steady laminar, forced convective heat transfer at a front stagnation point due to a step change in the wall temperature was investigated. In these analyses, the solutions for the non-steady energy equations were obtained by a Laplace transform technique. Examining the results reported in [1–3] for the final decay to the steady state reveals an uncertainty in mathematical form which may be attributed to the pitfalls of the approximation method used or the drawback of the lack of uniqueness in their solution methods. Therefore, the paradox of how the non-steady heat transfer process approaches steady state remains to be answered, and the present investigation is aimed in this direction.

In Ref. [1], Cess and Sparrow determined two appropriate asymptotic solutions, valid for small and large time. The large time solution is expressed in a perturbation of separable type with the integer power series in terms of the Laplace transform variable p. It is, therefore, the inverse transform of this series solution that has no physical meaning in the real time domain so that an approximate solution is constructed to fit the two asymptotic solutions in the Laplace plane. The final decay of the instantaneous wall heat flux to the steady state obtained from their results and expressed in the present notation may be given by:

$$\frac{q_w(\tau)}{q_{w,ss}} = 1 + \frac{A}{\sqrt{\tau}} e^{-a_1\tau} + B\tau\, e^{-a_2\tau} + \ldots, \qquad \text{for } \tau \gg 1 \tag{1}$$

where the constants A, B, a_1 and a_2 are functions of the Prandtl number. In order to avoid the inherent difficulty encountered in [1] when the large time solution is expressed in terms of the power series expansion of the Laplace transform parameter p, Chao and Jeng [2] have used the perturbation transformation coupled with Meksyn's boundary layer variable to obtain a solution for large time in the Laplace plane. It is

$$\frac{\bar{q}_w}{q_{w,ss}} = \frac{\sum_{n=0}^{\infty} b_n p^n}{1 + \sum_{n=1}^{\infty} c_n p^n} \tag{2}$$

where the b_n's and c_n's are constants depending on the Prandtl number. By ignoring terms involving second and higher degree in p, (the justification for this approximation is that when τ becomes large, p becomes small), they obtained

$$\frac{q_w(\tau)}{q_{w,ss}} = 1 + \left(\frac{b_0}{c_1} - \frac{b_1}{c_1^2}\right) e^{-\tau/c_1}. \tag{3}$$

Undoubtedly, if the terms of second degree in p appearing in (2) are retained, a different form of inverse transform will result. In order to overcome the drawback of the lack of uniqueness as described in the large time approximation presented in [2], Chen and Chao [3] applied the technique proposed in [4] to analyze the non-steady heat transfer of laminar boundary layers in wedge flow. To avoid obtaining the large time solution, they introduced the parameter γ in the series expansion valid for small time, and expanded the solution in terms of $(p + \gamma)$ rather than p for small time (large p). The inverse transform of this series solution is, in turn, to match the known steady state solution and the value of γ is then determined for the corresponding Prandtl number. For the wedge angle of π, which corresponds to two-dimensional stagnation flow, the final decay of surface flux to the steady-state as determined from their result takes the form

$$\frac{q_w(\tau)}{q_{w,ss}} = 1 + \{\theta_0'(0)\}^{-1} \times$$
$$\left\{\left(\frac{Pr}{2\pi\tau}\right)^{\frac{1}{2}} \frac{1}{\gamma_w \tau} + \sum_{n=1}^{\infty} \frac{u_n'(0)(2Pr\gamma_w)^{n/2}(\gamma_w\tau)^{n/2-1}}{\Gamma[n/2]} \left[1 - \frac{n/2 - 1}{\gamma_w \tau}\right]\right\} e^{-\gamma_w \tau} \tag{4}$$

where $u_n'(0)$, $\theta_0'(0)$ and γ_w are functions of the Prandtl number. Inspection of (1), (3) and (4) reveals that the steady state solution is shown to be approached in an exponential manner, but with a different weighting function which may or may not be a function of time.

In this paper, the problem of final decay to the steady state of the title problem is re-investigated without resorting to the Laplace transform technique. The thermal response characteristic in two-dimensional and axisymmetrical stagnation flow is shown to be exponential decay with time of perturbations from the steady state value. The characteristic factors in the exponents arise from the solution of an eigenvalue problem in ordinary linear differential equations. Two methods are proposed to determine eigenvalues for Prandtl numbers ranging from 0.01 to 100. The first method is a Karman-Pohlhausen integral technique which will be used to calculate the lowest eigenvalue. The second method is a finite difference technique by use of an electronic digital computer. An example of matching the present result to the small time solution obtained previously in [2] is given.

2. Fundamental equations

Assuming steady, incompressible flow with constant properties and negligible dissipation (small Mach number of the free stream), the governing boundary layer equations in the neighborhood of stagnation are

$$\text{Continuity:} \quad \frac{\partial(x^i u)}{\partial x} + \frac{\partial(x^i v)}{\partial y} = 0, \tag{5}$$

where $i = 0, 1$, respectively, for the two-dimensional and axisymmetrical stagnation flow.

$$\text{Momentum:} \quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = K^2 x + \nu\frac{\partial^2 u}{\partial y^2}, \tag{6}$$

where K is a constant which is related to the velocity just outside the boundary layer near the stagnation point. For example, K has a value of $2U_\infty/R$ for a cylinder with radius R.

Energy: $$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}. \tag{7}$$

The boundary conditions for the velocity field are

$$\begin{aligned} &u(x, y) = v(x, y) = 0 \quad &&\text{for } y = 0,\\ &u(x, y) = U_1 &&\text{for } y \to \infty, \end{aligned} \tag{8}$$

where U_1 designates the velocity at the edge of the boundary layer. The initial and boundary conditions for the temperature field are

$$T(x, y, t) = T_\infty \qquad \text{for } t \leqq 0, \tag{9a}$$

$$T(x, y, t) = T_\infty + (T_w - T_\infty)1(t) \quad \text{for } t > 0,\ y = 0, \tag{9b}$$

$$T(x, y, t) = T_\infty \qquad \text{for } y \to \infty. \tag{9c}$$

The continuity equation can be satisfied by introducing the stream function ψ, such that

$$u = \left(\frac{L}{x}\right)^i \frac{\partial\psi}{\partial y}, \qquad v = -\left(\frac{L}{x}\right)^i \frac{\partial\psi}{\partial x}, \tag{10}$$

where L is a representative length. The x and y coordinates will now be transformed by writing

$$\xi = \frac{Kx^2}{2u_\infty}\left(\frac{x}{L\sqrt{2}}\right)^i, \qquad \eta = y\left(\frac{2^i K}{\nu}\right)^{\frac{1}{2}}. \tag{11}$$

We further introduce a new variable $f(\xi, \eta)$ such that

$$\psi = \left(\frac{\nu K}{2^i}\right)^{\frac{1}{2}}\left(\frac{x}{L}\right)^i xf. \tag{12}$$

From equation (10), we find that

$$u = Kx\frac{\partial f}{\partial\eta}, \quad v = -(2^i\nu K)^{\frac{1}{2}} f \tag{13}$$

and the momentum equation (6) becomes

$$f''' + ff'' + \frac{1}{2^i}(1 - f'^2) = 0 \tag{14}$$

where the primes denote differentiation with respect to η, while f is a function of η only. The boundary conditions are

$$f(0) = f'(0) = 0 \text{ and } f'(\infty) = 1. \tag{15}$$

To transform the energy equation (7), we further introduce a dimensionless temperature,

$$\theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{16}$$

and a dimensionless time parameter,

$$\tau = \frac{2^{i}Kt}{Pr}. \tag{17}$$

It follows that

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial\eta^2} + Prf\frac{\partial\theta}{\partial\eta} - 2Prf'\xi\frac{\partial\theta}{\partial\xi}. \tag{18}$$

Along the stagnation line, the last term in equation (18) vanishes ($\partial\theta/\partial\xi = 0$), which is the convective contribution parallel to the surface. It is a valid approximation to neglect that term for the stagnation point problem. The initial and boundary conditions become

$$\begin{aligned} &\theta(\eta, \tau) = 0 && \text{for} \quad \tau \leqq 0, \\ &\theta(\eta, \tau) = 1(\tau) && \text{for} \quad \eta = 0, \tau > 0, \\ &\theta(\eta, \tau) = 0 && \text{for} \quad \eta \to \infty, \tau > 0. \end{aligned} \tag{19}$$

A complete solution of the problem involves solving equations (14) and (18) with their respective boundary conditions.

3. Asymptotic solution for two dimensional stagnation flow when τ is large

A solution for large τ is now sought in the form of a perturbation from the steady state $\theta_0(\eta)$. Therefore, let

$$\theta(\eta, \tau) = \theta_0(\eta) + \theta_1(\eta, \tau). \tag{20}$$

Upon substitution of (20) into (18) with $\partial\theta/\partial\xi = 0$, there results

$$\theta_0'' + Prf\theta_0' = 0 \tag{21}$$

with

$$\theta_0(0) = 1 \text{ and } \theta_0(\infty) = 0, \tag{21b}$$

and

$$\theta_1'' + Prf\theta_1' = \frac{\partial\theta_1}{\partial\tau} \tag{22a}$$

with

$$\begin{aligned} &\theta_1(\eta, \tau) = 0 \quad \text{for} \quad \eta = 0, \\ &\theta_1(\eta, \tau) = 0 \quad \text{for} \quad \eta \to \infty. \end{aligned} \tag{22b}$$

The initial condition with regard to time defines the particular "small time" solution to which the present large time solution may be joined, and will therefore remain unspecified. A procedure of integrating the steady state solution (21) and some numerical results have been given by Squire [5], Merk [6] and Meksyn [7]. The latter two authors showed that

$$-\theta_0'(0) = \frac{1}{\int_0^\infty \exp\left\{-Pr\int_0^\eta f\,d\eta\right\}d\eta} = \frac{0.6608Pr^{\frac{1}{3}}}{C(Pr)} \tag{23}$$

where

$$C(Pr) = 1 + 0.11583Pr^{-\frac{1}{3}} + 0.0441Pr^{-\frac{2}{3}} + 0.00118Pr^{-1} + 0.00020Pr^{-\frac{4}{3}} - 0.0031Pr^{-\frac{5}{3}} + \dots .$$

Here some coefficients in $C(Pr)$ have been corrected by Chao and Jeng [2].

If we assume that the perturbation dimensionless temperature $\theta_1(\eta, \tau)$ is separable so that

$$\theta_1(\eta, \tau) = \Phi(\eta)T(\tau), \tag{24}$$

then from (22) and (24), it becomes immediately evident that $T(\tau)$ must be of the form

$$T(\tau) = \exp\{-\lambda\tau\}$$

where λ is a constant. It is apparent that if λ is a positive and real value, then θ_1 will experience exponential decay to zero as $\tau \to \infty$. In the next section, a quantitative discussion is made concerning λ. The function $\Phi(\eta)$ therefore satisfies

$$\Phi''(\eta) + Prf\Phi'(\eta) + \lambda\Phi(\eta) = 0 \tag{25a}$$

with

$$\Phi(0) = \Phi(\infty) = 0. \tag{25b}$$

Since the temperature external to the thermal boundary layer is independent of time, we expect the time-dependent perturbations to be confined to the thermal boundary layer. Thus we expect that $\Phi(\eta)$ approaches zero exponentially as $\eta \to \infty$. For some specific values of λ, we shall be able to find that Φ, if they exist, possess this characteristic. It is necessary to exclude the solutions of Φ for other values of λ which approach zero algebraically as $\eta \to \infty$. In this sense, the solution of (25) poses an eigenvalue problem for the parameter λ. The analogue criterion has also been used by Kelly [8] in obtaining the asymptotic solution for large time in the non-steady stagnation flow due to a step change in free stream velocity.

3.1 *Discussion of the eigenvalue* λ

In this section, we would like to show that a perturbation of the form (24) decays exponentially to zero as time goes to infinity, thus implying stability of the steady state. It is therefore necessary to show that all possible eigenvalues must be positive real numbers. Let us introduce a new function defined by

$$Y(\eta) = \Phi(\eta)\exp\left\{\frac{Pr}{2}\int_0^\eta f\,d\eta\right\} \tag{26}$$

into (25); one then obtains

$$Y'' - \left(\frac{Pr}{2}f' + \frac{Pr^2}{4}f^2\right)Y = -\lambda Y \tag{27a}$$

with

$$Y(0) = Y(\infty) = 0. \tag{27b}$$

Because f and f' are both positive, the form of (27a) and the boundary conditions on Y fulfill the conditions given by Titchmarsh [9] for the existence of a discrete set of eigenvalues, λ_n ($n = 1, 2, \ldots$), where λ_n tends to infinity as n tends to infinity. It can also be seen that the linear operator on the left-hand side of (27a) is self-adjoint, and since the boundary conditions (27b) are regular, the theory of linear ordinary differential equations guarantees that the eigenvalues are real. In order to show that the eigenvalues are positive, (27a) is multiplied by Y^*, which is the complex conjugate of Y, and integrated with respect to η from 0 to ∞, to obtain

$$\int_0^\infty Y''Y^* d\eta + \int_0^\infty \left\{\left(\lambda - \frac{Pr}{2}f' - \frac{Pr^2}{4}f^2\right)YY^*\right\} d\eta = 0. \tag{28}$$

After an integration by parts of the first term in (28), equation (28) becomes

$$Y^*Y'|_0^\infty - \int_0^\infty Y'Y'^* d\eta + \int_0^\infty \left\{\left(\lambda - \frac{Pr}{2}f' - \frac{Pr^2}{4}f^2\right)YY^*\right\} d\eta = 0 \tag{29}$$

with $Y(0) = Y^*(0) = 0$. The asymptotic form of Y' for large η is required in order to demonstrate the vanishing of the integrated term and the convergence of the integral as $\eta \to \infty$. The asymptotic solution for $f(\eta)$ when $\eta \to \infty$ is given by Rosenhead [10] as

$$f \simeq \eta - 0.6479 \tag{30}$$

With this asymptotic form of f, (27a) becomes

$$Y'' + \left\{\lambda - \frac{Pr}{2} - \frac{Pr^2}{4}(\eta - 0.6479)^2\right\} Y = 0 \quad \text{for } \eta \gg 1 \tag{31}$$

which is a linear differential equation with irregular singularities at infinity, the solutions of which can be asymptotic with respect to the independent variable. The quantity $(\lambda - Pr/2)$ in (31) may be neglected as compared to $Pr^2 (\eta - 0.6479)^2/4$. The dominant term for the asymptotic solution of (31) valid for large η is then given by

$$Y \sim \left[\frac{Pr}{2}(\eta - 0.6479)\right]^{-\frac{1}{2}} \exp\left\{-\frac{Pr}{2}(\eta - 0.6479)^2\right\}, \tag{32}$$

and then

$$Y' \sim \left[\frac{Pr}{2}(\eta - 0.6479)\right]^{\frac{1}{2}} \exp\left\{-\frac{Pr}{2}(\eta - 0.6479)^2\right\}, \tag{33}$$

and when $\eta \to \infty$, $Y' \to 0$. With the integrated term set to zero, (29) may therefore be written as

$$\int_0^\infty \left\{Y'Y'^* + \left[\frac{Pr}{2}f' + \frac{Pr^2}{4}f^2 - \lambda\right]YY^*\right\} d\eta = 0. \tag{34}$$

Since f' and $Y'Y'^*$ are positive throughout, the above integral can not be equal to zero for negative values of λ. Therefore the eigenvalues are positive, if they exist. Thus we have shown that all possible eigenvalues are positive real numbers.

3.2. *Numerical determination of the eigenvalues and eigenfunctions for two dimensional stagnation flow*

Equation (25) will be used to calculate the eigenvalues. First, the Karman-Pohlhausen integral method will be employed to calculate the approximate lowest eigenvalues and then a more accurate determination of the first two eigenvalues are obtained by using an electronic digital computer for Prandtl numbers ranging from 0.01 to 100.

To apply the Karman-Pohlhausen technique, (25a) is first integrated to give

$$[\Phi']_0^\infty + Pr[f\Phi]_0^\infty - Pr\int_0^\infty \Phi f' \, d\eta + \lambda \int_0^\infty \Phi \, d\eta = 0, \tag{35}$$

and by the use of boundary condition (25b) and also $\Phi'(\infty) = 0$, λ can be expressed as

$$\lambda = \frac{\Phi'(0) + Pr\int_0^\infty \Phi f' \, d\eta}{\int_0^\infty \Phi \, d\eta}. \tag{36}$$

In order to integrate (36), the forms of $f'(\eta)$ and $\Phi(\eta)$ should be assumed. We assume that $f'(\eta)$ can be expressed approximately [8] in the form

$$f'(\eta) = 1 - 1.3075 \exp\{-1.7799\eta\} + 0.3075 \exp\{-3.5599\eta\} \tag{37}$$

where the numerical constants are chosen to satisfy the boundary conditions (15) and give (37) the best fit to known data for $f'(\eta)$. We now assume $\Phi(\eta)$ to be of the form

$$\Phi(\eta) = \eta \, e^{-\beta\eta^2}. \tag{38}$$

The above function satisfies $\Phi(0) = \Phi'(\infty) = \Phi''(0) = 0$, where the last condition is obtained by evaluating (25a) at $\eta = 0$. In assuming $\Phi(\eta)$ to take the form (38), we have normalized the eigenfunctions by imposing the conditions $\Phi'(0) = 1$. Obviously this normalization will not affect the final results. The incorrect exponential behavior of (37) and (38) as $\eta \to \infty$ is assumed to be unimportant in the evaluation of the integral in (36). An additional integral relation, which is required to determine the constant β appearing in (38), is obtained by multiplying (25a) by Φ' and integrating to get

$$\int_0^\infty \Phi''\Phi' \, d\eta + Pr\int_0^\infty f(\Phi')^2 \, d\eta + \lambda \int_0^\infty \Phi\Phi' \, d\eta = 0. \tag{39}$$

After integration by parts and applying the boundary conditions, (39) becomes

$$2Pr\int_0^\infty f(\Phi')^2 \, d\eta = 1 \tag{40}$$

which does not involve λ. We may thus determine β by substituting the differential form of (35) and f obtained from the integration of (37) into (40). Once β is determined, λ may be calculated from (36). The results of β and λ for various Prandtl numbers are tabulated in Table 1.

TABLE 1

Values of β and the first eigenvalue obtained by the integral method for two-dimensional stagnation flow.

Pr	β	λ_1
0.01	0.004621	0.01923
0.1	0.04050	0.1781
0.7	0.2277	1.066
1.0	0.3078	1.456
10	1.894	9.463
100	9.865	50.72

TABLE 2

The first and second eigenvalues for two-dimensional stagnation flow.

Pr	λ_1	λ_2
0.01	0.01951	0.03931
0.1	0.1840	0.3758
0.7	1.1130	2.363
1.0	1.518	3.266
10.0	9.520	22.377
100.0	49.70	121.77

More accurate first and second eigenvalues and the corresponding eigenfunctions for various Prandtl numbers are obtained by using the IBM 360 digital computer. The numerical integration of (14) and (25) is achieved by employing a fourth order Runge-Kutta numerical integration procedure incorporated onto a computer program which may continuously assign the value of λ. The increment of λ varies from Prandtl number to Prandtl number. The final determination of the eigenvalues, λ, is done by observing the resulting behavior of Φ, which has a smooth exponential decay at infinity; this criterion led to the eigenvalues shown in Table 2.

It is noticed that the first eigenvalue obtained by the Karman-Pohlhausen method is a very accurate approximation. Comparing with the exact value, they differ by less than 4% for the range of Prandtl numbers studied. The first two eigenfunctions for $Pr = 0.7$ are plotted in Fig. 1. Data for other eigenfunctions for the cases of Pr selected in this study may be found in [11].

One may see from Fig. 1 that Φ_1 exhibits non zero behavior between its end points. Because Φ has no zero in its assumed form in (38), the eigenvalues in Table 1 obtained by the Karman-Pohlhausen integral technique should be an approximation to the lowest eigenvalue.

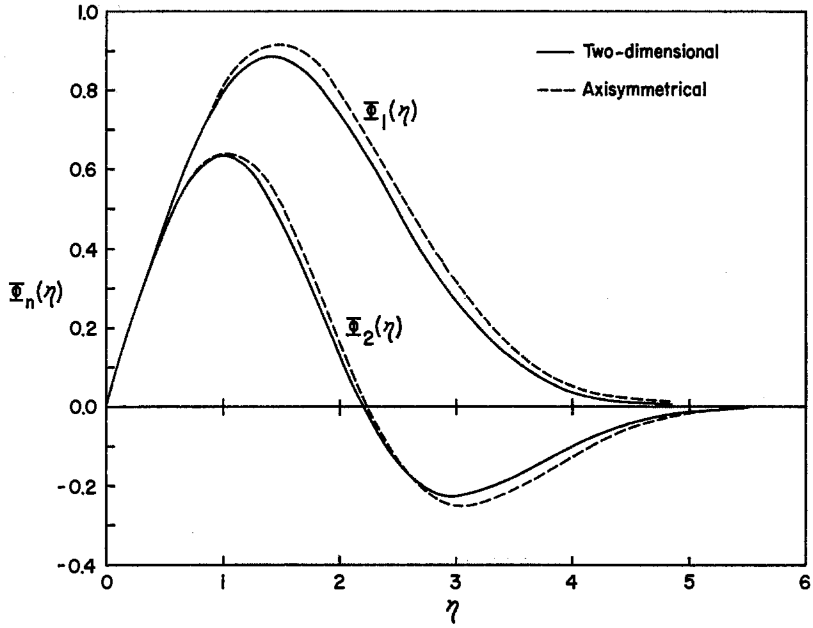

Figure 1. First two eigenfunction for $Pr = 0.7$.

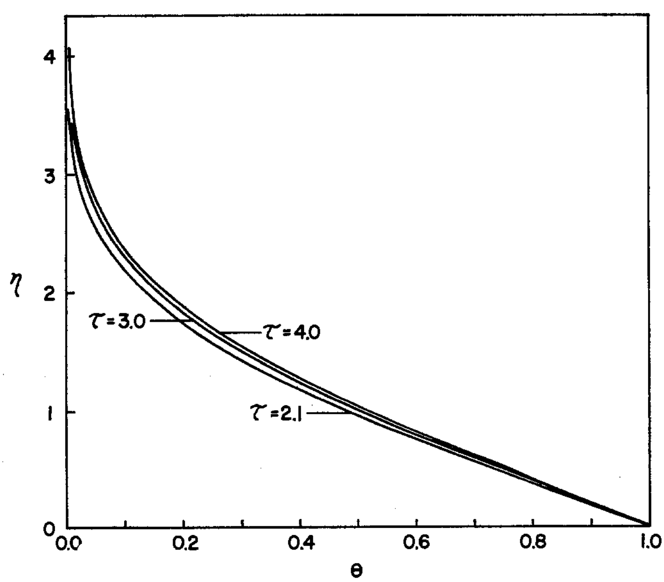

Figure 2. Final development of non-steady temperature field for $Pr = 0.7$.

With the values of λ_n's and Φ_n's obtained, the desired nonsteady temperature field for large τ can be expressed as

$$\theta(\eta, \tau) = \theta_0(\eta) + \sum_{n=1}^{\infty} A_n \Phi_n(\eta, \lambda_n)\,\mathrm{e}^{-\lambda_n \tau}. \tag{41}$$

The constants A_n's will depend, in some way, upon the initial growth of the thermal boundary layer. The determination of A_n's by matching the value of (41) and its derivative to the small time solution of [2] will now be discussed.

The instantaneous and steady-state wall flux are given respectively by

$$q_w = -k \left.\frac{\partial T}{\partial y}\right|_{y=0} = -k(T_w - T_\infty)\left(\frac{K}{\nu}\right)^{\frac{1}{2}} \theta'(0, \tau) \tag{42}$$

and

$$q_{w,ss} = -k(T_w - T_\infty)\left(\frac{K}{\nu}\right)^{\frac{1}{2}} \theta_0'(0). \tag{43}$$

Therefore

$$\frac{q_w}{q_{w,ss}} = 1 + \frac{1}{\theta_0'(0)} \sum_{n=1}^{\infty} A_n\,\mathrm{e}^{-\lambda_n \tau}, \text{ for large } \tau. \tag{44}$$

A simple numerical match of (44) to the small time solution of [2], i.e.,

$$\begin{aligned}\frac{q_w}{q_{w,ss}} = \frac{C(Pr)}{0.6608 Pr^{\frac{1}{3}}} \Bigg\{ &\frac{1}{\Gamma(\frac{1}{2})} \tau^{-\frac{1}{2}} + 0 + 0 + \frac{aPr}{8}\tau - \frac{Pr}{16\Gamma(\frac{5}{2})}\tau^{\frac{3}{2}} + 0 \\ &+ \frac{a^2 Pr(Pr+2)}{128\Gamma(\frac{7}{2})}\tau^{\frac{5}{2}} - \frac{aPr(Pr+2)}{128\Gamma(4)}\tau^3 + \frac{Pr(Pr+4)}{512\Gamma(\frac{9}{2})}\tau^{\frac{7}{2}} \\ &- \frac{a^3 Pr[1 + Pr(45Pr - 1)]}{512\Gamma(5)}\tau^4 + \ldots \Bigg\}\end{aligned} \tag{45}$$

will be attempted here to determine A_1 and A_2. Both the values of $q_w/q_{w,ss}$ and its first derivative are matched at an appropriate dimensionless time, τ_j, at which the two asymptotic solutions (44) and (45) are joined. Values of τ_j given in Table 2 of [2] were used for the matching. The results are shown in Table 3. With the evaluation of A_n's, the final development of the transient temperature field becomes completely determined and is given by (41). They are illustrated in Fig. 2 for $Pr = 0.7$. The results show that at $\tau = 4$, considerable development of the transient temperature field has already taken place and all data points fall on or within one percent of the steady-state values. For engineering calculations, this

TABLE 3

Values of the constants A_1 and A_2

Pr	0.1	0.7	1.0	10	100
A_1	−0.2615	−0.5633	−0.6940	−1.6928	− 3.829
A_2	0.01015	1.649	2.572	8.701	35.270

time may be interpreted as the time required to reach steady-state. It is also seen from Fig. 2 that the transient temperature profile approaches steady-state uniformly. This is in contrast with the unsteady heat transfer for flow over a flat plate in which, for large τ, the departure from the steady-state is ultimately concentrated near the wall [13]. It is worthwhile to mention that the present technique may be also applied to obtain the solution for large time due to a step change in wall heat flux.

4. Axially symmetric stagnation flow

As the analysis in this case is quite similar to that for two-dimensional flow, we shall merely present the results supplemented with a few brief comments when deemed desirable. The momentum equation for this case is given by (14) with $i = 1$. Using the same transformation variables as defined in (20) and following a similar procedure, one obtains the non-steady temperature profile as expressed in (41). The steady wall temperature gradient is given by

$$-\theta_0'(0) = \frac{0.6011 Pr^{\frac{1}{3}}}{Ca(Pr)} \tag{46}$$

where

$$Ca(Pr) = 1 + 0.08460 Pr^{-\frac{1}{3}} + 0.02352 Pr^{-\frac{2}{3}} + 0.00911 Pr^{-1} + 0.00021 Pr^{-\frac{4}{3}} + 0.00089 Pr^{-\frac{5}{3}} + \ldots .$$

In complete analogy to the steps described in the proceeding sections, the eigenvalues and eigenfunctions are obtained from (25) by using an electronic computer. No attempt was made to use the Karman-Pohlhausen integral method for this case. As before, algebraic decay of the solution at infinity is excluded in favor of exponential decay. The first two eigenvalues are tabulated in Table 4 for the various Prandtl numbers.

TABLE 4

The first and second eigenvalues for axisymmetrical stagnation heat transfer

Pr	λ_1	λ_2
0.1	0.18038	0.3701
0.7	1.0507	2.267
1.0	1.418	3.109
10	8.339	19.950

The corresponding eigenfunctions are plotted in Fig. 1 for the case of $Pr = 0.7$. The eigenfunctions for other values of the Prandtl number are given in Ref. [11].

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